# The Birthday Paradox

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## The Problem

- The birthday paradox, also known as the birthday problem, states that in a random group of 23 people, there is about a 50 percent chance that two people have the same birthday.
- There are multiple reasons why this seems like a paradox. New sources will be added it to this ever growing presentation.

1. Scientific American proposes the same problem, creating another paradox, by stating:

   a) Every one of the 253 combinations (of two persons) has the same odds, $p = 0.99726027$, of not being a match, *they say*.
   b) If you calculate (364/365)ˆ253, you'll find there's a 49.952 percent chance that all 253 comparisons contain no matches, *they say*.

Assuming a year of 365 days only

## 3 people only

- Why go for 23, before understanding the problem with only 3 persons?
- Define $A_i$ the event where the $i$ person will not share the same birthday with nobody and the negation by $\bar{A}_i$.
- We have 3 combinations (of two persons) $A_1A_2$, $A_1A_3$, $A_2A_3$.
- $P(A_1A_2)$ denotes the probability of both $A_1$ and $A_2$ being TRUE.
- If $A_1A_2$ and $A_1A_3$ both not being a match (TRUE), then $A_2A_3$ can be FALSE (share the same birthday).
- BUT: $P(A_1A_2A_3|G_3) = P(A_2A_3|G_3)$, so problem solved (one combination), why asking for $p^3$?

## Bayes settings

- We start by declaring the sample space, $G_3$, being a group of 3 persons, and here the negation $\bar{A}_i$ depends on $G_3$.
- The Problem is symmetric, so finding $P(A_3|G_3)$ will determine $P(A_2|G_3)$ as well.
- $P(\bar{A}_1\bar{A}_2\bar{A}_3|G_3) = 1/365^2$ and this can be generalized for any group
- $P(\bar{A}_1\bar{A}_2\bar{A}_3|G_3) = P(\bar{A}_1|\bar{A}_2\bar{A}_3G_3) \cdot P(\bar{A}_2\bar{A}_3|G_3) \Rightarrow P(\bar{A}_2\bar{A}_3|G_3) = 1/365$ (1)
- $P(A_1A_2A_3|G_3) = P(A_1|A_2A_3G_3) \cdot P(A_2A_3|G_3) = P(A_2A_3|G_3)$, by simple logic: if $A_3$ is TRUE and $A_2$ is TRUE, then the first person can't share the same birthday with nobody else in $G_3$, so the first probability is one. Also holds to be true for any $n > 2$:

$$P(A_1A_2...A_n|G_n) = P(A_1|A_2 \ldots A_nG_n) \cdot P(A_2...A_n|G_n)$$

## Bayes calculations

- (2): $P(\bar{A}_2|\bar{A}_3G_3) = 1/2$, by knowing that the third person shares the *birthday*, we have a 50% chance that will mach either of the remaining persons.
- (1)+(2): $P(\bar{A}_2\bar{A}_3|G_3) = P(\bar{A}_2|\bar{A}_3G_3) \cdot P(\bar{A}_3|G_3)$ so we get $P(\bar{A}_3|G_3) = 2/365$ or $P(A_3|G_3) = 363/365$
- Actually $P(A_2|A_3G_3) = 364/365$ so only by knowing that the third person is alone, will impose that the other persons will have *that claimed chance* of being alone (not matching the remaining person, or a $G_2$ problem)
- Finally, $P(A_2A_3|G_3)) = P(A_2|A_3G_3)P(A_3|G_3) = 363 \cdot 364/365^2$

## Another solution

- Even though not recommended in general, Bayes being preferred, sampling all possible different birthdays of 23 persons can pave the way.

- The number of ordered arrangements of 23 days taken from 365 unlike days is $365!\,/(365-23)!$
- Not to confuse with the number of ways of selecting 23 different days! Not looking for $\binom{365}{23}$, because
- We consider the total number of cases $365^{23}$, so we get $P(A_1A_2...A_{23}|G_{23}) = 364 \cdot 363 \cdots 343/365^{22} \approx 0.4927$

## Mistakes explained

- $p = P(A_1A_2)$ has different meanings for different groups of people, so claiming that $p = 364/365$ it is false in general, but TRUE in $G_2$

- Same mistakes can be found almost everywhere, the difference in calculations are luckily close (49.952% vs. 49.27%)
- Can not mix results from $G_2$ to $G_n$, for $n > 2$
- We might guess that the intention was to calculate the probabilities of having 2 different days as a pair and impose it to all pairs. But $(A_2, A_3)$ or any other pair of events are not independent!
- Now imagine there were 366 persons in the group. It is clear that $P(A_1 A_2 ... A_{366} | G_{366}) = 0$, but $(364/365)^{66795} > 0$

## Remarks

- Google returned the selected papers, no judgmental sampling was intended. The author has not been involved in a dispute with the editors of the mentioned papers
- The views expressed in this article are those of the author.
- This is an updated report using the latest information and tries to adapt the main idea using more information as they appear,
- actively pursuing error-correction by creating criticisms of both existing ideas and new proposals.
- To create an annotation, select any text and then select the Annotate button either in a private or public group.